Symbolic dynamics of a driven time-delay system

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18-05-2023

Motivation

Ryan, Keane & Amann; Chaos 30, 023121 (2020)

- While time delay and periodic drive have both been heavily studied individually, the combination of the two features has received less attention.
- Such systems can be difficult to study analytically, and must be studied numerically.
- **Goal:** To study the combination of time delay and periodic drive in a system so simple it can be studied analytically.

The GZT model

Ghil, Zaliapin & Thompson; Nonlin. Proc. Geophys, 15(3), 417-433, 2008.

GZT model:

$$\dot{h}(t) = -\tanh\left[\kappa h\left(t-\tau\right)\right] + b\cos\left(2\pi t\right)$$

with initial condition

$$h\left(t\right) \text{ for } t \in \left[-\tau, 0\right] \text{ is known.}$$

Keane, Krauskopf & Postlethwaite; SIAM J. Appl. Dyn. Syst., 14(3):1229, 2015.

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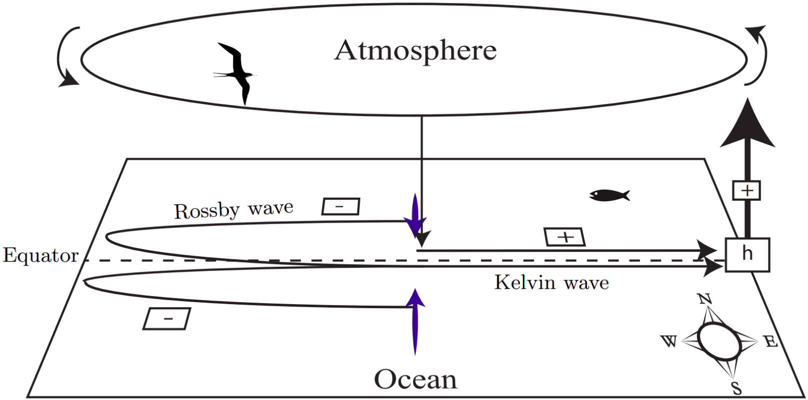

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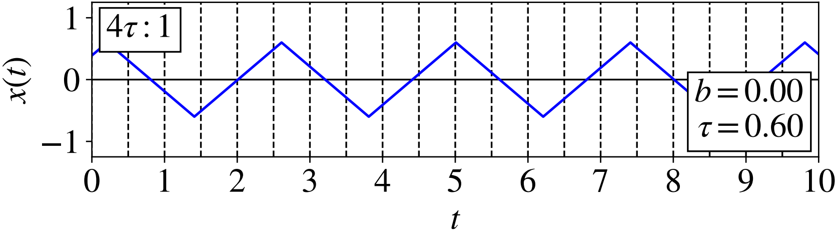
$$x\left(0\right) \text{ and } \left\{t \in \left[-\tau, 0\right] : x\left(t\right) = 0\right\} \text{ are known.}$$

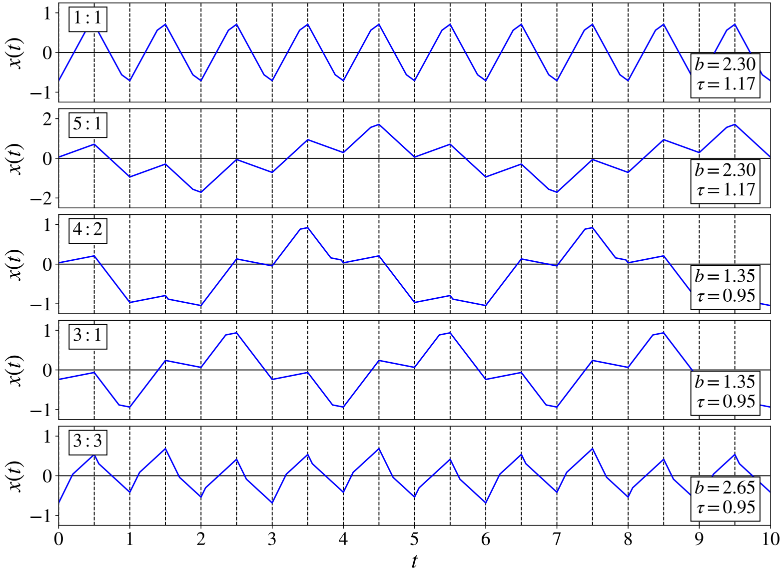
Dynamics of the unforced system

$$\dot{x}(t) = -\text{sgn}\left[x\left(t - \tau\right)\right] + b\ \text{sgn}\left[\cos\left(2\pi t\right)\right]$$



P : R solution: A solution with period P and R crossings from $x < 0$ to $x > 0$ per period.

Dynamics of the forced system



Dynamics in the (b, τ) plane

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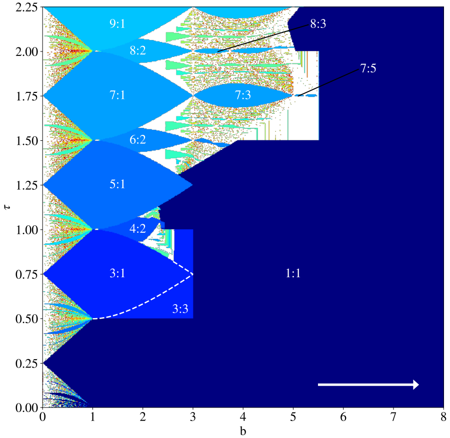

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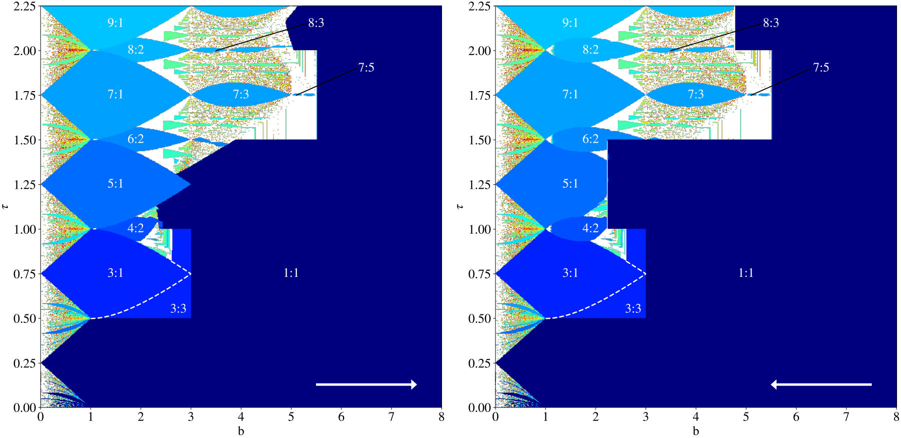

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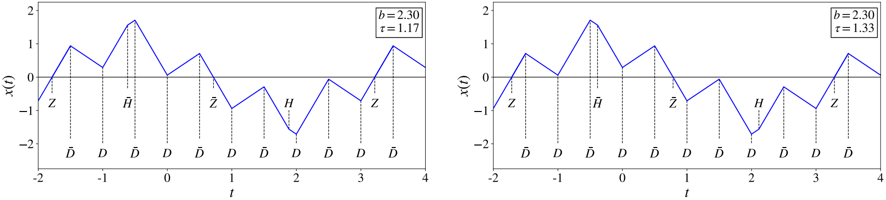
Six distinct event symbols. A $P : R$ solution has a sequence which repeats after $2P + 4R$ events.



Stable 5:1 solution for $\tau < \frac{5}{4}$: $\left[Z, \bar{D}, D, \bar{H}, \bar{D}, D, \bar{D}, \bar{Z}, D, \bar{D}, H, D, \bar{D}, D\right]$

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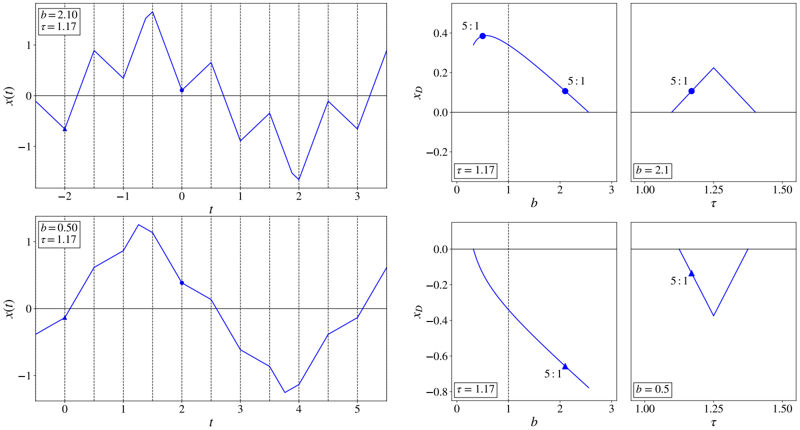
Stability:

- Each sequence generates a Poincaré map where the solution represented by the sequence is a fixed point.
- Stability of the fixed point $\equiv$ Stability of the solution.

Bifurcations of the 5 : 1 solution



Stable 5 : 1 $\left[Z, \bar{D}, D, \bar{H}, \bar{D}, D, \bar{D}, \bar{Z}, D, \bar{D}, H, D, \bar{D}, D\right]$

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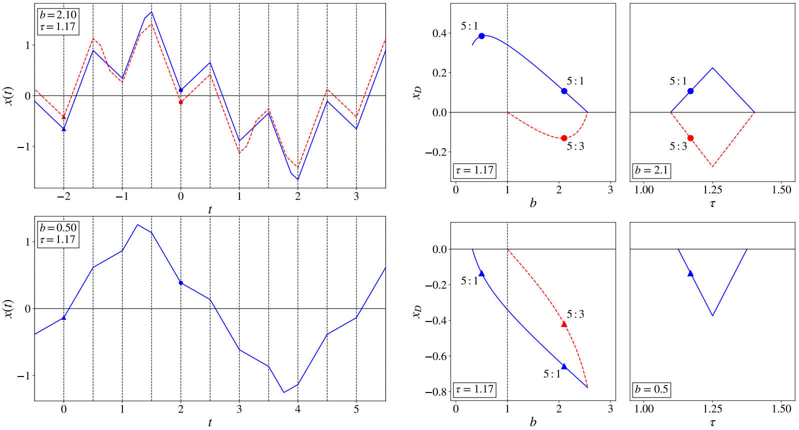

$5:1\left[Z,\bar{D},D,\bar{H},\bar{D},D,\bar{D},\bar{Z},D,\bar{D},H,D,\bar{D},D\right]$

$5:3\left[Z,\bar{D},\bar{H},H,D,\bar{H},\bar{D},\bar{Z},D,Z,\bar{D},\bar{Z},D,H,\bar{H},\bar{D},H,D,Z,\bar{D},\bar{Z},D\right]$

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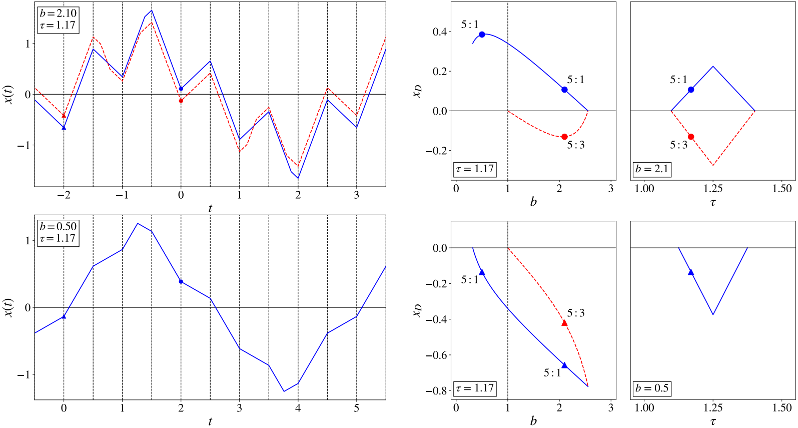
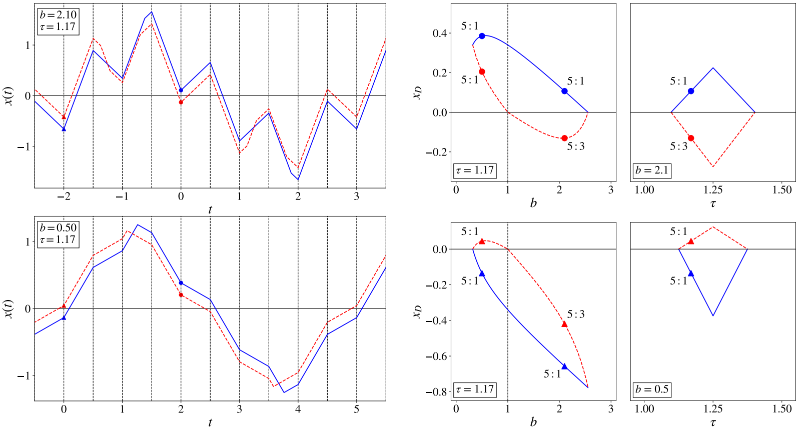


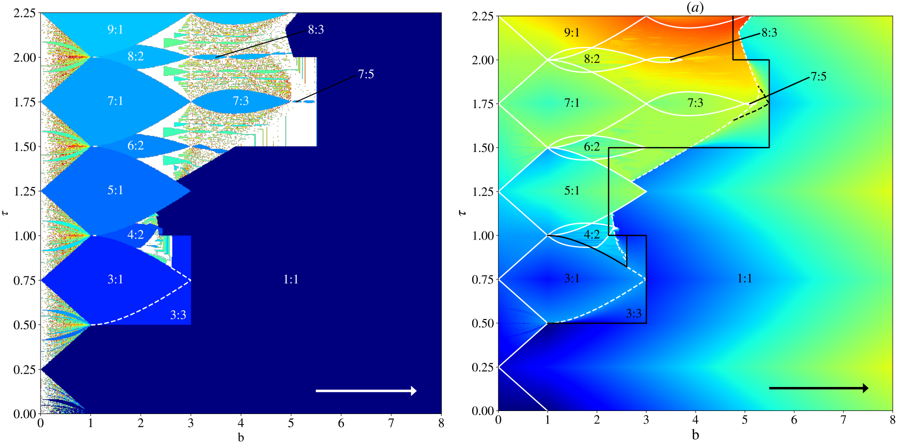
$$\mathbb{B}_P(x_D) = \begin{cases} -\left(\frac{b-1}{b+1}\right) x_D + \frac{2b}{b+1} - \frac{b+|P-4\tau|}{2} & x_D \geq 0 \\ \left(\frac{4b}{b^2-1} - \frac{b-1}{b+1}\right) x_D + \frac{2b}{b+1} - \frac{b+|P-4\tau|}{2} & x_D < 0 \end{cases}$$



Stable 5 : 1 $\left[Z, \bar{D}, D, \bar{H}, \bar{D}, D, \bar{D}, \bar{Z}, D, \bar{D}, H, D, \bar{D}, D\right]$

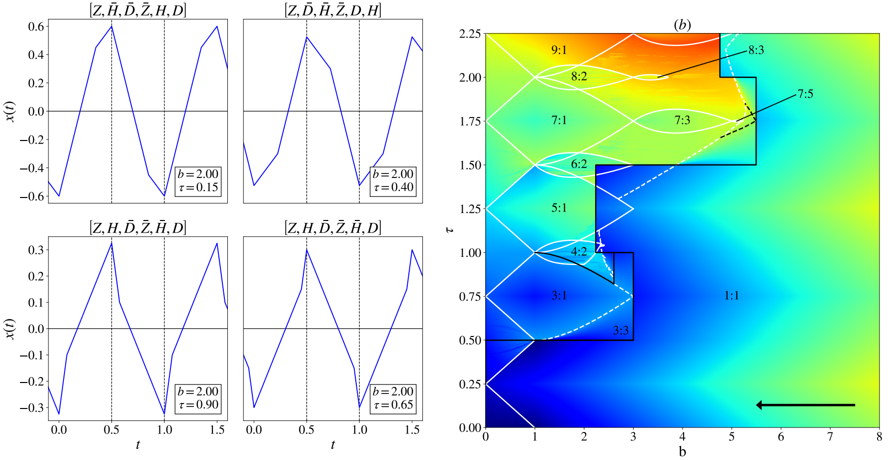
Unstable 5 : 1 $\left[Z, D, \bar{D}, D, \bar{H}, \bar{D}, D, \bar{Z}, \bar{D}, D, \bar{D}, H, D, \bar{D}\right]$

Bifurcation curves in the (b, τ) plane



Solid white curves are BCSN bifurcations.
Solid black curves are T and BCT bifurcations.

Bifurcations of the 1 : 1 solution



$$b_{\text{bif}}(n) = \frac{1}{\cos\left(\frac{\pi(n-1)}{2n-1}\right)} - (-1)^{n-1}, \; n = \lceil 2\tau \rceil$$

Comparison to GZT system

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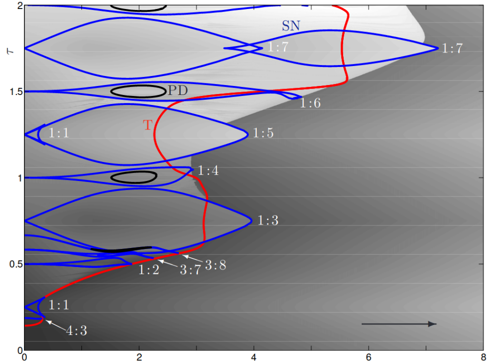

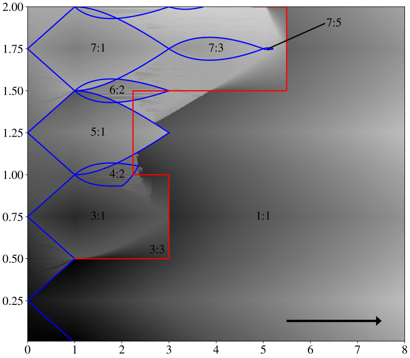

Our system

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Outlook

GZT system (all smooth):

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Our system (all non-smooth):

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Acknowledgements

Research supported by the Irish Research Council Employment-Based Postgraduate Programme and McAfee LLC under grant EBPPG/2018/269.

Attendance supported by the UCC School of Mathematical Sciences Staff Development & Research Support Fund.

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